Average power loss per unit length due to Eddy current heating of a thin walled infinitely long round beam pipe

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Abstract

I want to derive the formula for the average power loss/unit length of a thin walled, infinitely long round beam pipe from Eddy currents that Moritz[2, 3] stated but did not prove. I have listed the assumptions that I have made in order to derive the formula. Unfortunately, some of the assumptions required for deriving the formula are suspect because of the "infinitely" long beam pipe requirement. Therefore, *caveat emptor* with its use!

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I. THEORY

I want to derive the average power loss per unit length of an infinitely long, thin walled round beam pipe from Eddy currents due to an oscillating B-field that is normal to the pipe's axis. The analysis is not that straight forward! But there is a reference, Haus[1], from which I will use to start my analysis.

The geometry that I will analyze is shown in Fig. 1. The B-field is in the y direction. Unfortunately, this means that there is no natural coordinate system for both $\boldsymbol{E}$ and $\boldsymbol{B}$ in this geometry. I will use the cylindrical coordinate system: (r, θ, z) for $\boldsymbol{E}$ and Cartesian coordinate sytem: (x, y, z) for $\boldsymbol{B}$.

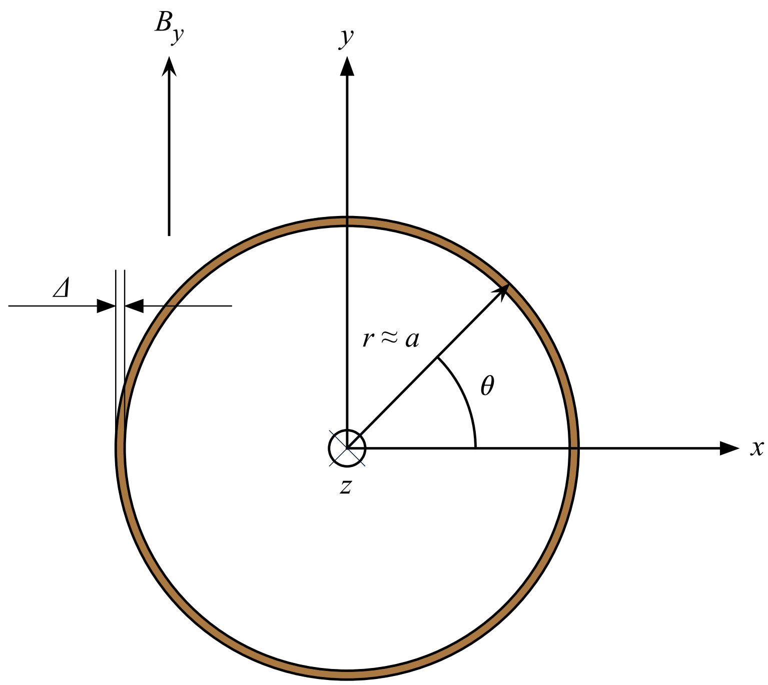

FIG. 1. This shows the transverse cross section of the round beam pipe of radius a and thickness Δ. The dipole field, B_y, is pointing in the y direction. Shell currents flow in the $\pm z$ direction.

In this analysis, I will assume that the beam pipe is infinitely long so that I don't need to take care of the edge effects. But here's more assumptions that I have to make in order to derive the formula:

1. The shell currents flow in both the $+z$ and $-z$ directions. A shell current that flows in the $+z$ direction must return in the $-z$ direction. Since the beam pipe is infinitely

long, there has to be perfect shorts at the ends of the beam pipe that is infinitely far away to allow for the current paths to close!

2. The wall thickness $\Delta \ll \delta$ where δ is the skin depth.

3. When I apply Ohm's law, taking into account point (1), I only have the z component of the current density, $\boldsymbol{j}$, to worry about

$$j_z(r,\theta) = \sigma E_z(r,\theta) \tag{1}$$

where σ is the conductivity, and I have explicitly noted the dependence of j_z and thus E_z on their location. I then use point (2) that states that $\Delta \ll \delta$, this means that j_z is constant in the cross section of the wall. Let k_z be the surface current density then I have at $r = a$

$$\begin{aligned} & k_z(r=a,\theta) = j_z(r=a,\theta)\Delta \\ \Rightarrow \quad & k_z(r=a,\theta) = \sigma\Delta \times E_z(r=a,\theta) \end{aligned} \tag{2}$$

4. The wall has the same permeability, μ_0, as vacuum, i.e. it is not magnetic. This requirement ensures that the external field is not disturbed by the beam pipe because it is essentially transparent to the B-field.

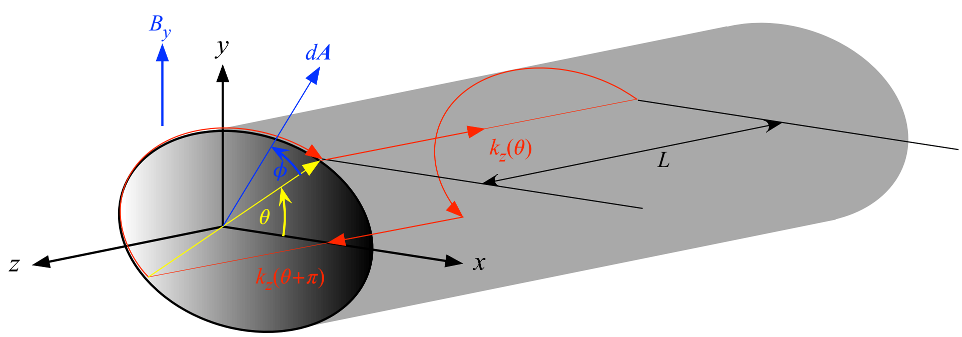

FIG. 2. The red lines show the path that I have chosen for the surface current density k_z to flow. The path encloses half the surface of the beam pipe. By symmetry arguments $k_z(\theta+\pi) = -k_z(\theta)$. The normal vector that defines the surface $d\boldsymbol{A}$ is at an angle ϕ w.r.t. θ.

I will start with the integral form of Faraday's law which is

$$\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \int_A \boldsymbol{B} \cdot d\boldsymbol{A} \tag{3}$$

The integral path, in red, is shown in Fig. 2. The arcs of the path span half the beam pipe, i.e. each arc has length πa.

Therefore, by using Eq. 3, I can calculate $k_z(\theta)$ shown in the Fig. 2 where I can exploit the symmetry of the path because $k_z(\theta + \pi) = -k_z(\theta)$. Thus Eq. 3 becomes

$$\left. \begin{aligned} -\frac{L}{\sigma\Delta}[k_z(\theta) - k_z(\theta + \pi)] &= -\frac{2L}{\sigma\Delta} k_z(\theta) \\ &= -\frac{d}{dt} \int_A \boldsymbol{B} \cdot d\boldsymbol{A} \end{aligned} \right\} \tag{4}$$

where I have used Eq. 2, and the "$-$" sign on the lhs comes from the $k_z(\theta)$ current flowing in the opposite direction w.r.t. $\hat{\boldsymbol{z}}$ in Fig. 2. There are no contributions to the lhs from the arcs because the surface current density is always normal to the arcs from point (1) above.

For the rhs, which has the integral over the enclosed area defined by the path, I can write $\boldsymbol{B}_z$ in (x, y) coordinates as

$$\boldsymbol{B}_z = \begin{pmatrix} 0 \\ B_z \end{pmatrix} \tag{5}$$

and $d\boldsymbol{A}$ in (x, y) coordinates

$$d\boldsymbol{A} = (L \times a\ d\phi) \begin{pmatrix} \cos(\theta + \phi) \\ \sin(\theta + \phi) \end{pmatrix} \tag{6}$$

Thus the integral

$$\int_A \boldsymbol{B} \cdot d\boldsymbol{A} = aB_z L \int_0^\pi \sin(\theta + \phi)\ d\phi = 2aB_z L \cos\theta \tag{7}$$

I can then substitute the above into Eq. 4 to obtain the surface current density, k_z in terms of $\dot{B}_y$ and θ

$$\begin{aligned} \frac{2L}{\sigma\Delta} k_z(\theta) &= 2a\dot{B}_z L \cos\theta \\ \Rightarrow \quad k_z(\theta) &= a\sigma\Delta \times \dot{B}_z \cos\theta \end{aligned} \tag{8}$$

which is independent of L (as it should be).

A. Power loss per unit length

From Eq. 8, the current dI_z that flows into a strip that is $a\,d\theta$ wide is

$$dI_z = k_z(\theta)a\,d\theta \tag{9}$$

The resistivity of the beam pipe is $\rho = 1/\sigma$ (units of $\Omega \cdot \mathrm{m}$) and the resistance of a beam pipe of length L and cross sectional area $\Delta \times a\,d\theta$ is

$$R = \rho\frac{L}{\Delta \times a\,d\theta} = \frac{L}{\sigma\Delta \times a\,d\theta} \tag{10}$$

Thus, the instantaneous power loss is simply

$$\left.\begin{aligned} dP &= dI_z^2 R = (k_z(\theta)a\,d\theta)^2 \times \frac{L}{\sigma\Delta \times a\,d\theta} = \frac{k_z^2(\theta)L}{\sigma\Delta}a\,d\theta \\ &= a^3 L\sigma\Delta\dot{B}_y^2\cos^2\theta\;d\theta \end{aligned}\right\} \tag{11}$$

Finally, I can integrate θ from 0 to 2π to obtain the <u>instantaneous</u> power loss per unit length of the beam pipe and it is

$$\boxed{P/L = a^3\sigma\Delta \times \dot{B}_y^2 \int_0^{2\pi} \cos^2\theta\;d\theta = a^3\pi\sigma\Delta \times \dot{B}_y^2} \tag{12}$$

which is the same equation that Moritz showed in his report (eq. 37) [2] and his slides [3].

1. *Average power loss per unit length*

I will adopt a sinusoidal form for B_y in order to calculate the average power loss per unit length. Let

$$B_y(t) = B_0 \sin\omega t + \text{DC offset} \tag{13}$$

where ω is the ramp frequency and B_0 is the magnitude of the B-field. The DC offset is irrelevant because I will differentiate the above to get

$$\dot{B}_y(t) = B_0\omega\cos\omega t \tag{14}$$

which I can substitute into Eq. 12 and average over 1 period

$$\left.\begin{aligned} P_{\text{ave}}/L &= a^3\pi\sigma\Delta \times \frac{\omega}{2\pi}\int_0^{2\pi/\omega} B_0^2\omega^2\cos^2\omega t\;dt \\ &= a^3\pi\sigma\Delta \times \frac{B_0^2\omega^2}{2} \end{aligned}\right\} \tag{15}$$

or in the more familiar form, where I replace $\omega = 2\pi f$ to get

$$\boxed{P_{\text{ave}}/L = 2\pi^3 a^3 f^2 \sigma\Delta \times B_0^2} \tag{16}$$

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[1] Haus, Hermann A., and J.R. Melcher. Electromagnetic fields and energy (Massachusetts Institute of Technology: MIT OpenCourseWare. `http://ocw.mit.edu`, 2020.

[2] G. Moritz. Eddy currents in accelerator magnets. `https://cds.cern.ch/record/1335027/files/103.pdf`, June 2009.

[3] G. Moritz. Eddy currents in accelerator magnets. `https://cas.web.cern.ch/sites/cas.web.cern.ch/files/lectures/bruges-2009/moritz-new.pdf`, June 2009.